LECTURE V

Bi-presymplectic separability of Stäckel systems

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Dual Poisson-presymplectic pair

- How to relate Hamiltonian vector fields and inverse-Hamiltonian vector fields in degenerated cas? $(X_H = \Pi dH, \quad \Omega X^H = dH)$

Definition

Dual Poisson-presymplectic pair of corank m on M we call a pair (Π, Ω) such that

1. $\ker \Pi = Sp\{dc_i, \quad i = 1, ..., m\}$
2. $\ker \Omega = Sp\{Z_i, \quad i = 1, ..., m\}$
3. $Z_i(c_j) = \delta_{ij}, \quad i, j = 1, ..., m$
4. The following partition of unity holds on TM, respectively T^*M :

$$I = \Pi\Omega + \sum_{i=1}^{m} Z_i \otimes dc_i, \qquad I = \Omega\Pi + \sum_{i=1}^{m} dc_i \otimes Z_i.$$

- **Observation.** On any symplectic leave S of Π : $\left(\Pi_{|S}\right)^{-1} = \Omega_{|S}$.

Dual Poisson-presymplectic pair

- Let (Π,Ω) be a dual pair and $X_F=\Pi dF,\ \ \Omega X^F=dF$, then

$$dF=\Omega X_F+\sum_{i=1}^{m}Z_i(F)dc_i,\qquad X_F=X^F-\sum_{i=1}^{m}X^F(c_i)Z_i.$$

- Observe that for Poisson algebra given by a dual pair (Π,Ω), although $X_F\neq X^F$, but

$$\begin{aligned}\{F,G\}^{\Omega}:=\Omega(X_F,X_G)&=\langle\Omega X_F,X_G\rangle=\langle\Omega X^F,X_G\rangle=\langle dF,\Pi dG\rangle\\&=\{F,G\}_{\Pi}.\end{aligned}$$

- For any dual pair (Π,Ω) :

$$L_{X_F}\Pi=0,\quad L_{X^F}\Omega=0,\quad L_{Z_i}\Pi=0,\quad L_{Z_i}\Omega=0,\quad [Z_i,Z_j]=0.$$

- Non-uniqueness of dual pairs.

Dual Poisson-presymplectic pair

Theorem

Let (Π, Ω) *be a dual pair with* $\ker \Pi = Sp\{dc_i\}$ *and* $\ker \Omega = Sp\{Z_i\}$. *Let*

$$\Omega' = \Omega + \sum_{i=1}^{m} dc_i \wedge df_i,$$

then (Π, Ω') *is again dual pair, with* $\ker \Omega' = Sp\{Z_i + \Pi df_i\}$, *provided that*

$$\Pi(df_i, df_j) + Z_j(f_i) - Z_i(f_j) = 0.$$

Let

$$\Pi' = \Pi + \sum_{i=1}^{m} Z_i \wedge X_i, \qquad \Omega X_i = dF_i,$$

then (Π', Ω) *is again dual pair, with* $\ker \Pi' = Sp\{dc_i + dF_i\}$, *provided that*

$$\Omega(X_i, X_j) + X_j(c_i) - X_i(c_j) = 0.$$

Dual Poisson-presymplectic pair

- **Examples.**
- $2n$-dimensional phase space $M=\mathbb{R}^{2n}$ with nondegenerated canonical dual pair:

$$\pi=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial p_i},\quad \omega=\sum_{i=1}^{n}dp_i\wedge dx_i,\quad \pi\omega=\omega\pi=I,$$

$$\pi=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix},\quad \omega=\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}.$$

- Extend $M\to\mathcal{M}=M\times\mathbb{R}^m$ with extra coordinates $(c_1,\ldots,c_m)$. Then, on $\mathcal{M}$

$$\Pi=\begin{bmatrix}0 & I & 0\\ -I & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \Omega=\begin{bmatrix}0 & -I & 0\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\ker\Pi=Sp\{dc_i\},\quad \ker\Omega=Sp\left\{\frac{\partial}{\partial c_i}\right\},\quad \frac{\partial}{\partial c_i}(c_j)=\delta_{ij},$$

Dual Poisson-presymplectic pair

- $$I = \Pi\Omega + \sum_{i=1}^{m} \frac{\partial}{\partial c_i} \otimes dc_i.$$
- Gauge freedom for $m = 1$.
- For any Hamiltonian vector field $X_F = \Pi dF$, such that $\frac{\partial F}{\partial c} = 0$,

$$\Pi' = \Pi + \frac{\partial}{\partial c} \wedge X_F, \qquad \ker \Pi' = d(c + F)$$

  is dual to Ω.
- For any function f

$$\Omega' = \Omega + dc \wedge dF, \qquad \ker \Omega' = \frac{\partial}{\partial c} + \Pi df$$

  is dual to Π.

D-compatibility

Definition

A Poisson tensor Π_1 is *d*-compatible with a Poisson tensor Π_0 if there exists a presymplectic form Ω_0, dual to Π_0, such that $\Omega_0 \Pi_1 \Omega_0$ is closed. Then, we say that the pair (Π_0, Π_1) is *d*-compatible with respect to Ω_0.

Definition

A closed two-form Ω_1 is *d*-compatible with a closed two-form Ω_0 if there exists a Poisson tensor Π_0, dual to Ω_0, such that $\Pi_0 \Omega_1 \Pi_0$ is Poisson. Then, we say that the pair (Ω_0, Ω_1) is *d*-compatible with respect to Π_0.

- For Π_0 nondegenerated:

$$d\text{-compatibility} \iff \text{ordinary compatibility}$$

- ωN-manifold case: (π_0, π_1) are *d*-compatible with respect to $\omega_0 = \pi_0^{-1}$ and (ω_0, ω_1), where $\omega_1 = \omega_0 \pi_1 \omega_0$, are *d*-compatible with respect to π_0.

D-compatibility

- For Π_0 degenerated:

$$d\text{-compatibility} \implies \text{ordinary compatibility}$$

- For implication $\Longleftarrow$ an additional assumption is required, i.e. the existence of some Ω_0, dual to Π_0, such that

$$\Omega_0(L_{Z_i}\Pi_1)\Omega_0 = 0, \qquad i = 1, ..., r.$$

- From above condition follows that

$$L_{Z_i}\Pi_1 = \sum_{j=1}^{m} W_{ij} \wedge Z_j$$

and hence, according to the results of Lecture III, if a pair (Π_0, Π_1) is d-compatible with respect to Ω_0, then Π_1 is projectible onto the foliation of Π_0 along the distribution $\mathcal{Z} = \ker \Omega_0$.

Bi-inverse-Hamiltonian chains

Theorem

Assume that there exists a pair of presymplectic forms (Ω_0, Ω_1) *d-compatible with respect to some* Π_0 *dual to* Ω_0*, both of rank* $2n$ *and co-rank* m *on* M*. Assume further, that they form bi-inverse-Hamiltonian chains of closed one-forms*

$$dH_i^{(k)} = \Omega_0 Y_{i+1}^{(k)} = \Omega_1 Y_i^{(k)}, \qquad i = 1, ..., n_k, \tag{5.1}$$

where $k = 1, ..., r$, $n_1 + ... + n_m = n$ *and each chain starts with a kernel vector field* $Y_0^{(k)}$ *of* Ω_0 *and terminates with a kernel vector field* $Y_{n_k}^{(k)}$ *of* Ω_1*. Then*

$$\Omega_0(Y_i^{(k)}, Y_j^{(s)}) = \Omega_1(Y_i^{(k)}, Y_j^{(s)}) = 0.$$

Bi-inverse-Hamiltonian chains

- Moreover, let
$$X_i^{(k)} = \Pi_0 dH_i^{(k)}$$
  which implies that
$$X_i^{(k)} = Y_i^{(k)} - \sum_{j=1}^{r} Y_i^{(k)}(H_0^{(j)})\, Y_0^{(j)},$$
  where $\Pi_0 dH_0^{(j)} = 0$. Then,
- $$\Pi_0(dH_i^{(k)}, dH_j^{(s)}) = 0, \qquad [X_i^{(k)}, X_j^{(s)}] = 0,$$
  so the chain defines a Liouville integrable system.

Any bi-inverse-Hamiltonian system (5.1) has quasi-bi-Hamiltonian representation on any leave of Π_0:

$$\begin{aligned}\Pi_0 dH_{i+1}^{(k)} &= \Pi_0\Omega_1 Y_i^{(k)} = \Pi_0\Omega_1\left(X_i^{(k)} + \sum_{j=1}^{m} Y_i^{(k)}(H_0^{(j)})\,Y_0^{(j)}\right)\\ &= \Pi_0\left(\Omega_1 X_i^{(k)} + \sum_{j=1}^{m} Y_i^{(k)}(H_0^{(j)})\,dH_1^{(j)}\right)\\ &= \Pi_0\Omega_1\Pi_0 dH_i^{(k)} + \sum_{j=1}^{m} Y_i^{(k)}(H_0^{(j)})\Pi_0 dH_1^{(j)}\end{aligned}$$

$$\Updownarrow$$

$$\Pi_D dH_i^{(k)} = \Pi_0 dH_{i+1}^{(k)} - \sum_{j=1}^{m}\alpha_{ij}^{(k)}\Pi_0 dH_1^{(j)} \tag{5.2}$$

where $\Pi_D = \Pi_0\Omega_1\Pi_0$ and $\alpha_{ij}^{(k)} = Y_i^{(k)}(H_0^{(j)})$.

Bi-inverse-Hamiltonian chains

- Π_D is Poissson as (Ω_0, Ω_1) are compatible.
- Moreover Π_D and Π_0 share the same Casimirs $\{H_0^{(k)}\}$, so (5.2) can be restricted to any leave S of Π_0 of dimension $2n$:

$$\pi_1 dh_i^{(k)} = \pi_0 dh_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} \pi_0 dh_1^{(j)},$$

where $\pi_0 = \Pi_{0|S}$, $\pi_1 = \Pi_{D|S}$, $h_i^{(k)} = H_{i|S}^{(k)}$, and we again landing in bi-Lagrangian distribution of ωN-manifold, considered in Lecture III.

Bi-inverse-Hamiltonian representation of Stäckel systems

- Separation relations on phase space M

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i,\mu_i)\left[\lambda_i^{r_k} + h^{(k)}(\lambda_i,n_k)\right] = \chi_i(\lambda_i,\mu_i), \qquad i = 1,\dots,n$$

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$$\Downarrow \quad \text{quasi-bi-Hamiltonian chains}$$

$$\pi_1 dh_i^{(k)} = \pi_0\left(dh_{i+1}^{(k)} - \sum_{j=1}^{m}\alpha_{ij}^{(k)} dh_1^{(j)}\right), \qquad \alpha_{ij}^{(k)} = V_i^{(k,j,n_j)}, \quad (5.3)$$

where

$$\pi_0 = \sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}, \quad \pi_1 = \sum_{i=1}^{n}\lambda_i\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}.$$

Bi-inverse-Hamiltonian representation of Stäckel systems

- Consider following symplectic forms on M

$$\omega_0 = \sum_{i=1}^{n} d\mu_i \wedge d\lambda_i, \qquad \omega_1 = \sum_{i=1}^{n} \lambda_i d\mu_i \wedge d\lambda_i.$$

- Observe that (π_0, ω_0) is a dual pair, $(\pi_0, \pi_1 = \pi_0\omega_1\pi_0)$ are d-compatible with respect to ω_0 and (ω_0, ω_1) are d-compatible with respect to π_0.
- Quasi-bi-Hamiltonian chains (5.3) have equivalent quasi-bi-inverse -Hamiltonian representation. Actually, multiplying (5.3) by ω_0 we get

$$\omega_1 x_i^{(k)} = \omega_0 \left(x_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} x_1^{(j)} \right),$$

where $x_i^{(k)} = \pi_0 dh_i^{(k)}$, $\omega_0 x_i^{(k)} = dh_i^{(k)}$.

- Lift: $M \to \mathcal{M}$, $(\lambda,\mu) \to (\lambda,\mu,c)$, $\dim \mathcal{M} = 2n+m$, $\omega_0 \to \Omega_0$, $\pi_0 \to \Pi_0$, $\ker \Omega_0 = Sp\{Y_0^{(k)}\}$, $\ker \Pi_0 = Sp\{dc_k\}$, (Ω_0, Π_0) dual pair.
- Similarly: $\omega_1 \to \Omega_D$, $\pi_1 \to \Pi_D$, $x_i^{(k)} \to X_i^{(k)}$, where $\ker \Omega_D = \ker \Omega_0$, $\ker \Pi_D = \ker \Pi_0$.
- Quasi-bi-inverse-Hamiltonian representation on $\mathcal{M}$:

$$\Omega_D X_i^{(k)} = \Omega_0 \left(X_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} X_1^{(j)} \right).$$

- Define a presymplectic two-form

$$\Omega_1 = \Omega_D + \sum_{k=1}^{m} dh_1^{(k)} \wedge dc_k$$

Bi-inverse-Hamiltonian representation of Stäckel systems

- and the set of vector-fields

$$Y_i^{(k)} = X_i^{(k)} + \sum_{j=1}^{m} \alpha_{ij}^{(k)} Y_0^{(j)}.$$

Theorem

On $\mathcal{M}$ differentials $dh_i^{(k)}$ form a bi-inverse-Hamiltonian chains

$$\begin{aligned}
\Omega_0 Y_0^{(k)} &= 0 \\
\Omega_0 Y_1^{(k)} &= dh_1^{(k)} = \Omega_1 Y_0^{(k)} \\
&\vdots \\
\Omega_0 Y_{n_k}^{(k)} &= dh_{n_k}^{(k)} = \Omega_1 Y_{n_{k-1}}^{(k)} \\
0 &= \Omega_1 Y_{n_k}^{(k)}, \qquad k = 1, ..., m,
\end{aligned}$$

where (Ω_0, Ω_1) are d-compatible with respect to Π_0.

Bi-inverse-Hamiltonian representation of Stäckel systems

- Let us compare the construction of bi-inverse-Hamiltonian repesentation with the construction of bi-Hamiltonian representation presented in Lecture IV.
- Extend the oryginal Hamiltonians

$$h_i^{(k)} \to H_i^{(k)} = h_i^{(k)} + \sum_{j=1}^{m} V_i^{(k,j,n_j)} c_j .$$

- Then on $\mathcal{M}$, vector fields $K_i^{(k)} = \Pi_0 dH_i^{(k)}$ form a bi-Hamiltonian chains

$$\begin{aligned}
\Pi_0 dH_0^{(k)} &= 0 \\
\Pi_0 dH_1^{(k)} &= X_1^{(k)} = \Pi_1 dH_0^{(k)} \\
&\vdots \\
\Pi_0 dH_{n_k}^{(k)} &= X_{n_k}^{(k)} = \Pi_1 dH_{n_{k-1}}^{(k)} \\
0 &= \Pi_1 dH_{n_k}^{(k)}
\end{aligned}$$

- where

$$\Pi_1 = \Pi_D + \sum_{j=1}^{m} K_1^{(j)} \wedge Y_0^{(j)}$$

  and (Π_0, Π_1) are d-compatible with respect to Ω_0.
- **Example.** Henon-Heiles
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$$\begin{aligned} h_1 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1 q_2^2, \\ h_2 &= \frac{1}{2}q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 + \frac{1}{4}q_1^2 q_2^2 + \frac{1}{16}q_2^2. \end{aligned}$$

- On $\mathbb{R}^5 \ni (q_1, q_2, p_1, p_2, c)$

$$\begin{aligned} \Omega_0 Y_0 &= 0 \\ \Omega_0 Y_1 &= dh_1 = \Omega_1 Y_0 \\ \Omega_0 Y_2 &= dh_2 = \Omega_1 Y_1 \\ 0 &= \Omega_1 Y_2, \end{aligned}$$

where $Y_1 = \Pi_0 dh_1 - q_1 Y_0$, $Y_2 = \Pi_0 dh_2 - \frac{1}{4} q_2^2 Y_0$ and

$$\Omega_0 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_1 = \begin{bmatrix} 0 & -\frac{1}{2}p_2 & -q_1 & -\frac{1}{2}q_2 & 3q_1^2 + \frac{1}{2}q_2^2 \\ \frac{1}{2}p_2 & 0 & -\frac{1}{2}q_2 & 0 & q_1 q_2 \\ q_1 & \frac{1}{2}q_2 & 0 & 0 & p_1 \\ \frac{1}{2}q_2 & 0 & 0 & 0 & p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 & -q_1 q_2 & -p_1 & -p_2 & 0 \end{bmatrix}.$$